① A set of continuous (smooth) functions S, is called uniformly complete, if $\underbrace{\forall \varepsilon > 0,}_{\text{given any } f \in C(M)}$ $\exists$ $\{f_j\}_{j=1}^{N} \subset S$ & $\{a_j\} \subset \mathbb{C}$ such that

$$\| f - \sum a_j f_j \|_\infty \leq \varepsilon .$$

Namely any continuous function can be uniformly approximated by the span of functions in S.

Recall, if G is compact, ϕ, ψ are two irreducible representations

$\phi : G \to GL(r, V)$, $\psi : G \to GL(r', V')$

$\left(\phi_{ik}(g)\right)_{1 \leq i,k \leq r}$ & $\left(\psi_{jl}(g)\right)_{1 \leq j,l \leq r'}$

Then $\langle \phi, \psi \rangle \doteq$ $\boxed{\int_G \phi_{ik} \overline{\psi_{jl}} \, du = \begin{cases} 0 & \text{if } \phi \nsim \psi \\ \dfrac{\delta_{ij}\delta_{kl}}{\dim(V)} & \phi \sim \psi . \end{cases}}$

$S \overset{\Rightarrow}{\doteq} \left\{ \sqrt{\dim \phi}\ \phi_{ik} \right\}_{\phi \in \hat{G}}$ forms an orthonormal set of (L^2-norm) smooth functions.

$\dim \phi \doteq \dim(V)$. ϕ irreducible

Defn: The representation ring is the ring of functions generated by S.

Theorem 1 (Peter-Weyl) S is uniformly complete.

In particular, any continuous function f, can by elements in the

[representation ring. ('ring' is Not needed!)]

I shall discuss a proof using PDE (eigenfunctions of the Laplace operator $\Delta \left(= -\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} (g^{ij} \sqrt{g} \frac{\partial}{\partial x^j} \right)$ locally.)

$\left\{ \begin{array}{l} ds^2 = g_{ij} dx^i dx^j \\ \text{is the Riemannian metric} \\ g = \det(g_{ij}) \end{array} \right.$

② This is from Ex 16, 20 of F. Warner's book. ch 6

Pf: (a) L^2-completeness of eigenfunctions.

(b) Uniform completeness of eigenfunctions

(c) eigenfunctions are inside the representation ring. In fact can be expressed as linear combinations of elements in S.

For (a), it requires some background on eigenvalues & eigenfunctions of elliptic operators

Ex 16, part (g) of F. Warner.

E.g. $G = S^1$, the completeness of $\{e^{in\theta}\}$, & Bessel inequality.

The keys are $\left\{ \begin{array}{l} \text{(i) } V_\lambda \text{ is finite dimensional} \quad V_\lambda \perp V_{\lambda'} \text{ if } \lambda \neq \lambda' \\ \text{(ii) } \{\lambda_i\} \text{ satisfies } \lambda_n \to \infty \text{ as } n \to \infty \end{array} \right.$

Sketch: Let $\boxed{G}$ be the solution operator.

Δ^{-1} — called Green operator $G(f) = u$ if $\Delta u = f$, $u \in (\ker \Delta)^\perp$ if $f \in (\ker \Delta)^\perp$

$\forall \alpha \in L^2(M)$. $\text{Span}\{u_i\}_{i=1}^k = \ker \Delta$. (For functions it is just $\left\{ \frac{1}{\sqrt{\mathrm{Vol}(M)}} \right\} \simeq \mathbb{C}$)

$$\alpha - \sum_{i=1}^{k} \langle \alpha, u_i \rangle u_i \perp \ker \Delta$$

$$\Rightarrow \exists \beta \in \ker(\Delta)^\perp, \quad G\beta = \alpha - \sum_{i=1}^{h} \langle \alpha, u_i \rangle u_i$$

$\xrightarrow{\quad} \left\{ \begin{array}{l} \text{By Hodge theory} \\ \alpha - H(\alpha) = \Delta \Omega \\ \beta := \Delta \Delta \Omega \\ G(\beta) = \Delta \Omega = \alpha - H(\alpha) \end{array} \right.$

$G \Delta \alpha = \Delta G \alpha = \alpha, \ \forall \alpha \in [\ker(\Delta)]^\perp$

$$\Rightarrow \underbrace{\left\| \alpha - \sum_{i=1}^{h} \langle \alpha, u_i \rangle u_i \right\|^2}_{G\beta = \alpha - \sum_{i=1}^{k} \langle \alpha, u_i \rangle u_i} = \left\| \underbrace{G}_{\text{unitary}} \underbrace{\left(\beta - \sum_{i=k+1}^{n} \langle \beta, u_i \rangle u_i \right)}_{} \right\|^2 \quad \boxed{n \geq k+1}$$

$G\beta = \alpha - \sum_{i=1}^{k} \langle \alpha, u_i \rangle u_i$,

$$\boxed{G u_j = \frac{1}{\lambda_j} u_j \qquad j \geq k+1}$$

Here $\{u_i\}_{i=1}^{\infty}$ eigenfunctions

unitary

For $$G\Big(\underbrace{\sum_{i=k+1}^{n} \langle\beta, u_i\rangle u_i}\Big) = \sum_{i=k+1}^{n} \Big\langle \beta, \frac{u_i}{\lambda_i}\Big\rangle u_i$$

$$= \sum \langle \beta, \underline{G(u_i)}\rangle u_i = \sum_{i=k+1}^{n} \langle \underline{G\beta}, u_i\rangle u_i$$

$$= \sum_{i=k+1}^{n} \Big\langle \alpha - \sum_{j=1}^{k}\langle \alpha, u_j\rangle \boxed{u_j, u_i}\Big\rangle u_i$$

$$= \sum_{i=k+1}^{n} \langle \alpha, u_i\rangle u_i$$ since $\langle u_j, u_i\rangle = 0$

$1 \le j \le k$

($k+1 \le i$ has $\lambda_i \neq 0$)

$$\Rightarrow \Big\|\alpha - \sum_{i=1}^{n}\langle\alpha, u_i\rangle u_i\Big\|^2 = \Big\|G\Big(\beta - \underbrace{\sum_{i=k+1}^{n}\langle\beta, u_i\rangle u_i}\Big)\Big\|^2$$

$$\le \boxed{\frac{1}{\lambda_{n+1}^2}}\Big\|\beta - \sum_{i=k+1}^{n}\langle\beta, u_i\rangle u_i\Big\|^2$$

$$\le \frac{1}{\lambda_{n+1}^2}\|\beta\|^2 \longrightarrow 0$$

$\boxed{\left\{ \lambda_n \to \infty \text{ as } n \to \infty \right.}$

given $\|\beta\|^2$ is fixed & $\lambda_{n+1} \longrightarrow +\infty$ as $n \to \infty$.

(b). Uses Sobolev embedding theorem:

Namely for $k >> 1$.

(*) $\|\alpha\|_\infty \le c\,\|(1+\Delta)^k \alpha\|$ for some $C = C(M, k)$.

Let $P_n\alpha = \sum_{i=1}^{n}\langle\alpha, u_i\rangle \underline{u_i}$ $\{u_i\}$ eigenfunctions of Δ with λ_i - eigenvalue.

$$\Delta P_n \alpha = \sum_{i=1}^{n}\langle\alpha, u_i\rangle \lambda_i u_i$$

$$= \sum\langle\alpha, \Delta u_i\rangle u_i = \sum\langle\Delta\alpha, u_i\rangle u_i$$

$$= P_n \Delta \alpha.$$

Now

$$\underbrace{\| \alpha - P_n\alpha \|_\infty}_{} \leq c \, \| \underbrace{(1+\Delta)^k (\alpha - P_n\alpha)}_{} \|$$

$$\underline{\varphi := (1+\Delta)^k \alpha}$$

$$= c \, \| (1+\Delta)^k \alpha - P_n (1+\Delta)^k \alpha \|$$

$$= \underbrace{c \| \varphi - P_n \varphi \|}_{} \to 0 \text{ as } n \to \infty, \text{ by (a).}$$

(c). Let $L_\sigma : G \to G$ be the left multiplication

It is easy to endow G with a left invariant metric

$\Rightarrow$ $L_\sigma : (G, ds^2)$ is an isometry.

Δ operator is a differential operator respects the isometries.

Namely. $\boxed{(L_\sigma f)(x) := f(\sigma x)}$ satisfies

[Δ invariant under the isometries]

$$\boxed{\Delta \circ L_\sigma = L_\sigma \circ \Delta}$$

precisely $[\Delta (L_\sigma f \; (x))] = (\Delta f)(\sigma x)$

[e.g. on $\mathbb{R}^n$ $(\Delta (f(x+a))) = (\Delta f)(x+a)$.]

Then Let $\{\varphi_i\}$ be a unitary basis of V_λ - the space of eigenfunctions with eigenvalue λ

$$\Delta \cdot \underbrace{L_\sigma \varphi_i}_{} = \underline{L_\sigma} \, \Delta \varphi_i = \lambda L_\sigma \varphi_i$$

$\Rightarrow L_\sigma \varphi_i \in V_\lambda \Rightarrow$

$$L_\sigma \varphi_i = \sum_{j=1}^{N} \varphi_j \boxed{G_{ji}(\sigma)} \qquad N = \dim(V_\lambda).$$

$$G_{ji}(\sigma) = \langle L_\sigma \varphi_i, \varphi_j \rangle = \int_G (L_\sigma \varphi_i) \overline{\varphi_j} \, d\mu$$

$$= \int_G \varphi_i(\sigma g) \overline{\varphi_j(g)} \, d\mu$$

Clearly it is continuous in σ from the above.

Moreover

$$L_{\sigma_1\sigma_2}\varphi_i = \sum \varphi_k G_{ki}(\sigma_1\sigma_2)$$

$$L_{\sigma_1}\circ L_{\sigma_2}\varphi_i = L_{\sigma_1}\left[\sum \varphi_j G_{ji}(\sigma_2)\right] = \sum \varphi_k G_{kj}(\sigma_1)\cdot G_{ji}(\sigma_2)$$

$$\Rightarrow \left(G_{ki}(\sigma_1\sigma_2)\right) = \left(G_{kj}(\sigma_1)\right)\cdot\left(G_{ji}(\sigma_2)\right).$$

Namely $\sigma \longrightarrow (G_{ji}(\sigma)) \in GL(N, V_\lambda)$ is a group homomorphism. Namely a representation of G in $GL(N, V_\lambda)$

$$\Rightarrow \quad G_{ji}(\sigma) \in \mathrm{Span}\{S\}$$

$$L_\sigma\varphi_i\Big|_{g=e} = \sum \varphi_j G_{ji}(\sigma)\Big|_{g=e}$$

⇙

On the other hand

$$\underbrace{(L_\sigma\varphi_i)(e)}_{\varphi_i(\sigma)} = \sum G_{ji}(\sigma)\underset{a_j}{\underset{\parallel}{\varphi_j(e)}}$$

$$\Rightarrow \boxed{\varphi_i(\sigma) = \sum a_j G_{ji}(\sigma)}$$

$$\Rightarrow \varphi_i(\sigma) \in \mathrm{Span}\{S\}$$

This completes (c). Hence we complete the proof of PW.

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③ Remark:

(a): $\varphi: G \to GL(r, V)$ is a group representation

$d\varphi: \mathfrak{g} \to \mathfrak{gl}(r, V)$ is a Lie algebra representation.

$$d\varphi([X\ Y]) = [\underset{\uparrow}{d\varphi(X)},\ d\varphi(Y)] = d\varphi(X)\cdot d\varphi(Y) - d\varphi(Y)\cdot ad_\varphi(X).$$

Lie algebra element in $\mathfrak{gl}(r, V)$.

$$\left[\begin{array}{llll} \varphi: & G_1 \to G_2 & & G_1 - \text{connected} \\ d\varphi: & \mathfrak{g}_1 \to \mathfrak{g}_2 & \Longrightarrow & d\varphi - \text{decides } \varphi \end{array}\right].$$

If G_1 is simply-connected, given $\pi: \mathfrak{g}_1 \to \mathfrak{g}_2$ $\exists$ a $\underset{d\varphi = \pi.}{\varphi}$ (existence).

Hence in the case G is simply-connected, the Lie group representation & Lie algebra representation are equivalent.

(b) $\phi_i: G \to GL(r_i\ V_i)$ $\qquad v_i \in V_i$

$$\phi_1 \oplus \phi_2: \quad G \to GL(V_1 \oplus V_2), \quad g \to \phi_1 \oplus \phi_2: (v_1, v_2) \to (\phi_1(g)(v_1), \phi_2(g)(v_2))$$

$$d(\phi_1 \oplus \phi_2): \quad \mathfrak{g} \to \mathfrak{gl}(V_1 \oplus V_2) \qquad X \to d(\phi_1 \oplus \phi_2): (v_1\ v_2) \to$$

$$(\underset{\in V_1}{d\phi_1(X)(v_1)},\ \underset{\in V_2}{d\phi_2(X)(v_2)})$$

$$\phi_1 \otimes \phi_2: \quad G \to GL(V_1 \otimes V_2)$$

$$g \to (\phi_1 \otimes \phi_2)(g)(v_1 \otimes v_2)$$

$$= \phi_1(v_1) \otimes \phi_2(v_2)$$

$$d(\phi_1 \otimes \phi_2)(X) = \frac{d}{dt}\Big|_{t=0} (\phi_1 \otimes \phi_2)(\exp(tX))(v_1 \otimes v_2)$$

$$= d\phi_1(x)(v_1) \otimes v_2 + v_1 \otimes d\phi_2(x)(v_2)$$

$$= \pi_1(X)(v_1) \otimes v_2 + v_1 \otimes \pi_2(v_2)$$

Abbreviated as

The action of $X \in \mathfrak{g}$ on $v_1 \otimes v_2$ is

$$\underbrace{Xv_1 \otimes v_2 + v_1 \otimes Xv_2}$$

$\left\{ \begin{array}{l} X \in \mathfrak{g} \text{ is viewed} \\ \text{as } \pi_1(X) \in \mathfrak{gl}(V_1) \\ \quad \pi_2(X) \in \mathfrak{gl}(V_2) \\ \pi_i : \mathfrak{g} \to \mathfrak{gl}(V_i) \end{array} \right.$

are two representations

(c)

Since $\phi^* : G \to GL(V^*)$, given $\phi : G \to GL(V)$

is defined as $\underbrace{\phi^*(g)(v^*)(w) \doteq v^*(g^{-1}w)}$

$$d\phi^*(X)(v^*)(w) = \frac{d}{dt}\Big|_{t=0} \phi^*(\exp(tX))(v^*)(w) = v^*\left(\phi\left(\frac{d}{dt}\Big|_{t=0} \exp(-tX)\right)(w)\right)$$

$$= v^*(d\phi(-X)(w)) = -v^*(d\phi(X)(w))$$

Namely if $\pi : \mathfrak{g} \to \underline{\mathfrak{gl}(V)}$ is the Lie algebra representation

$\pi^* : \mathfrak{g} \to \mathfrak{gl}(V^*)$ is defined as

$$\boxed{\pi^*(X)(v^*)(w) \doteq - v^*(\underline{\pi(X)(w)})}$$

Called $v^* \in V^*$

Contragredient representation.

(d). $\exists$ Another Peter-Weyl type theorem:

Let S_c be the character functions of all $\phi \in \hat{G}$

$\Rightarrow \forall f \in C(G)$ $f(gxg^{-1}) = f(x)$. namely f is a class function,

then f can be approximated by S_c.

See Pontryagin Theorem 34.

④ Lie algebra $\mathfrak{g}$ (regardless $\mathfrak{g} = \mathfrak{g}$ of G or any general/abstract $\mathfrak{g}$)

$\mathfrak{g}$. $\quad ad_X(Y) \doteq \underbrace{[X, Y]}_{}$

German

$B(X, Y) := tr(ad_X\, ad_Y)$ — called the $\underbrace{\text{Killing}}_{}$ form

Compute it for $\underbrace{gl(n, \mathbb{R})}_{}$ $gl(n, \mathbb{C})$ using

e_{ij} is the one $i \to \underbrace{\begin{pmatrix} 0 \cdots & \overset{j}{\downarrow} & 0 \\ - - - & 1 & \\ 0 & & 0 \end{pmatrix}}_{}$

$$\underbrace{[e_{ij}, e_{kl}]}_{e_{ij}\cdot e_{kl} - e_{kl}\cdot e_{ij}} = \underbrace{\delta_{jk}\, e_{il}}_{} - \underbrace{\delta_{li}\, e_{kj}}_{}$$

$X = X_{ij}\, e_{ij}$

$$tr(\boxed{ad_X}\boxed{ad_Y}) = \sum \langle ad_X\, ad_Y(e_{kl}),\ \underbrace{e_{kl}^*}_{} \rangle$$

$$= \langle ad_X (Y_{ij} (\delta_{jk} e_{il} - \delta_{li} e_{kj})) \quad e_{kl}^* \rangle$$

$$= \langle X_{st} Y_{ij} \delta_{jk}\, ad_{e_{st}} e_{il} - X_{st} Y_{ij} \delta_{li}\, ad_{e_{st}} e_{kj},\ e_{kl}^* \rangle$$

$$= \qquad \delta_{ti} e_{sl} - \delta_{ls} e_{it} \qquad\qquad \delta_{tk} e_{sj} - \underline{\delta_{js} e_{kt}}$$

$$n X_{ki} Y_{ik} - X_{ll} Y_{kk} - X_{kk} Y_{ii} + n X_{jl} Y_{lj}$$

$$X_{st} Y_{ij} \delta_{jk} \delta_{ti} \delta_{sk} \delta_{ll} = n X_{ki} Y_{ik}$$

$$X_{st} Y_{ij} \delta_{li} \delta_{js} \delta_{kk} \delta_{tl} = n X_{jl} Y_{lj}$$

$$= \boxed{2n\, tr(X Y^{tr}) - 2\, tr(X) tr(Y)}$$

Basic properties

(a) If $A \in Aut(\mathfrak{g})$ i.e. $\underbrace{\boxed{A([X, Y]) = [AX, AY]}}_{}$

$\Rightarrow \quad B(AX, AY) = B(X, Y)$

(b) If $a \in \underbrace{Der(\mathfrak{g})}_{}$ i.e. $\underbrace{a([X, Y]) = [aX, Y] + [X, aY]}_{}$

$$\Rightarrow \quad \underbrace{B(aX, Y) + B(X, aY)}_{} = 0$$

$$A(t)[X, Y] = [A(t)X, A(t)Y]$$

$$A'(t)[X, Y] = [A'(t)X, A(t)Y] + [A(t)X, A'(t)Y]$$

Note $\mathrm{Aut}(\mathfrak{g})$ has Lie algebra $\mathrm{Der}(\mathfrak{g})$

Since if $A(t) \in \mathrm{Aut}(\mathfrak{g})$ $A(0) = id$, $A'(0) = a$

$$\Rightarrow \quad \underbrace{a([X, Y]) = [aX, Y]}_{} + [X, aY]$$

On the other hand

$$\underbrace{\frac{d}{dt} e^{-ta}[e^{ta}X, e^{ta}Y]}_{} = \boxed{e^{-at}}\Big(-a\{[\underbrace{e^{ta}X}_{\bar X}, \underbrace{e^{ta}Y}_{\bar Y}]$$

$$+ \underbrace{[ae^{ta}X, e^{ta}Y]}_{[a\bar X, \bar Y]} + \underbrace{[e^{ta}X, ae^{ta}Y]}_{+[\bar X, a\bar Y]}\}\Big) = 0 \quad \text{if } a \in \mathrm{Der}(\mathfrak{g})$$

$$\Rightarrow \quad [e^{ta}X, e^{ta}Y] = e^{ta}([X, Y]) \quad \text{if } a \in \mathrm{Der}(\mathfrak{g})$$

$$\Rightarrow e^{ta} \in \mathrm{Aut}(\mathfrak{g}).$$

Proof of (a): $\boxed{A[X, Y] = [AX, AY]}$ $\forall Y$

$$A\,\mathrm{ad}_X(Y) = \mathrm{ad}_{AX} \circ A(Y)$$

$$\Rightarrow \quad \boxed{A\,\mathrm{ad}_X = \mathrm{ad}_{AX} \circ A} \quad \Rightarrow \quad \boxed{A\,\mathrm{ad}_X A^{-1} = \mathrm{ad}_{AX}}$$

$$\Rightarrow \quad \underbrace{B(AX, AY)}_{} = \mathrm{trace}\,(\mathrm{ad}_{AX}\,\mathrm{ad}_{AY})$$

$$= \mathrm{trace}\,(\underbrace{A\,\mathrm{ad}_X}_{} \cancel{A^{-1}} \cdot \cancel{A}\underbrace{\mathrm{ad}_Y}_{} A^{-1}) = \underbrace{B(X}_{}, Y)$$

The (b) follows from (a) since

$$\underbrace{B(e^{ta}X, e^{ta}Y)}_{} = B(X, Y). \Rightarrow \begin{array}{l} B(aX, Y) \\ + B(X, aY) = 0 \end{array} \qquad \square$$